M&C 2023 - The International Conference on Mathematics and Computational Methods Applied to Nuclear Science and Engineering · Niagara Falls, Ontario, Canada · August 13 – 17, 2023

P_1 Synthetic Acceleration and Convergence Analysis for the Solution of One-Speed Nonclassical Spectral S_N Equations in Slab Geometry

A. S. Silva[1], L. R. C. Moraes[1], R. C. Barros[1], and R. Vasques[2]

[1]Universidade do Estado do Rio de Janeiro,
Pós-graduação em Modelagem Computacional - IPRJ,
Rua Bonfim 25, 28625-570, Nova Friburgo, RJ, Brazil

[2]The Ohio State University,
Department of Mechanical and Aerospace Engineering
201 W. 19th Avenue, Columbus, OH 43210, USA

alansouza@iprj.uerj.br, lrcmoraes@iprj.uerj.br,
ricardob@iprj.uerj.br, richard.vasques@fulbrightmail.org

ABSTRACT

In classical transport theory, the distribution of free-path lengths of particles traveling through the media is exponential. However, in certain inhomogeneous media, the locations of scattering centers are spatially correlated, leading to a free-path length distribution that is not exponential. This has motivated the derivation of a generalized transport theory, also referred to as *nonclassical transport*, in which no assumption is made about the shape of the particle's free-path length distribution. In nonclassical transport, the free-path length of the particle is an independent variable in the (generalized) nonclassical transport equation. Recently, a Spectral Approach (SA) was introduced to deal with the free-path dependency on the nonclassical transport equation. In the SA, the nonclassical angular flux is represented as a truncated Laguerre series in the free-path variable. As an outcome, this representation generates a system of equations that have the form of classical transport equations. In this work, we use a synthetic acceleration scheme to speed up the iteration algorithm for the solution of the one-speed nonclassical spectral equations in the discrete ordinates formulation. The results of our numerical experiments indicate that the synthetic acceleration scheme is effective in reducing the number of iterations needed to obtain an accurate solution.

KEYWORDS: nonclassical transport, spectral approach, synthetic acceleration, Fourier analysis

1. INTRODUCTION

The study of particle transport has a wide range of applications in nuclear engineering, including reactor design, fuel cycle optimization, and the development of safeguards and monitoring systems [1]. In most of these applications, the particle's free-path length distribution (i.e., the distribution of the distances traveled by the particle between collisions) is given by an exponential, which is an inherent assumption of (classical) transport theory. Nevertheless, there are situations in which the free-path length distribution is not well-represented by an exponential. Examples of such situations include applications in photon transport in atmospheric clouds [2], neutron transport in certain types of nuclear reactors [3], and image rendering in computer graphics [4]. To mathematically address particle transport in such situations, a nonclassical transport equation has been derived [5]. This equation extends the phase-space to include the independent "memory" variable s, representing the distance, or free-path, traveled by the particle between interactions.

A Spectral Approach was introduced in [6] to deal with the free-path dependency on the nonclassical transport equation. In the SA, the nonclassical angular flux is represented as a truncated Laguerre series in the variable s. As an outcome, this representation generates a system of equations that have the form of classical transport equations and can therefore be solved by current deterministic algorithms. Numerical

results for one-speed slab-geometry problems were provided to validate the nonclassical spectral equations. Particularly, the authors have used the conventional discrete ordinates (S_N) formulation and the Diamond Difference method (DD) to deal with the angular and spatial dependencies on the angular flux, respectively. The resulting sweeping equations were used in conjunction with the Source Iteration (SI) scheme to generate numerical results for the classical scalar flux.

Similar to classical transport calculations (cf. [7]), the SI scheme demands a high number of iterations to achieve accurate results in nonclassical diffusive problems. Different techniques can be used to reduce the number of iterations needed in SI schemes, among which one can find synthetic acceleration methods. The first synthetic acceleration scheme proposed to solve the nonclassical spectral S_N equations was introduced in [8], based on the S_2 synthetic acceleration (S_2SA) method presented in [7] for the classical case.

The original contribution of the present work is a new acceleration technique for solving the nonclassical spectral S_N equations, which we call P_1 synthetic acceleration (P_1SA), based on the diffusion synthetic acceleration (DSA) approach for the classical transport equation [7]. Moreover, a Fourier convergence analysis has been developed to estimate the analytical spectral radius for SI, S_2SA, and P_1SA methods when applied to the nonclassical spectral S_N equations. By estimating the spectral radius of each method, we can infer if the iterative scheme is convergent and compare their convergence rate.

In the next section, we present the one-speed, slab-geometry nonclassical transport equation and briefly describe the Spectral Approach [6]. Moreover, a synthetic acceleration method for the one-speed slab-geometry nonclassical spectral S_N equations is described. In Section 3, we summarize the Fourier convergence analysis applied to the SI, S_2SA, and P_1SA methods. In Section 4, we present numerical results for two classes of test problems as well as the spectral radii obtained when using each of the three approaches. We close the paper in Section 5 with a discussion about the results and potential future work.

2. NONCLASSICAL FORMULATION AND SYNTHETIC ACCELERATION

In the classical theory of particle transport, the macroscopic total cross section σ_t is independent of the particle's direction-of-flight and of the particle's free-path s, defined as the distance traveled by the particle since its previous interaction. This assumption leads to an exponential free-path distribution function: $p(s) = \sigma_t e^{-\sigma_t s}$. In nonclassical transport, this assumption is relaxed; following [5,6,8], in this work we consider that σ_t is a function of s, such that [5]

$$p(s) = \sigma_t(s) e^{-\int_0^s \sigma_t(s')ds'}. \tag{1}$$

By considering the generalized $p(s)$ given in Eq. (1), we obtain a generalized version of the linear Boltzmann equation [5], referred to as nonclassical transport equation. In slab geometry, the steady-state, one-speed version of this equation can be written in an "initial value" form as [6]

$$\frac{\partial \Psi(x,\mu,s)}{\partial s} + \mu \frac{\partial \Psi(x,\mu,s)}{\partial x} + \sigma_t(s)\Psi(x,\mu,s) = 0, \quad s > 0,\ 0 < x < X, \tag{2a}$$

$$\Psi(x,\mu,0) = \frac{c}{2} \int_{-1}^{1} \int_{0}^{\infty} \sigma_t(s')\Psi(x,\mu',s')ds'd\mu' + \frac{Q(x)}{2}, \quad 0 < x < X, \tag{2b}$$

$$\Psi(0,\mu,s) = 0, \quad 0 < \mu \leq 1, s > 0, \tag{2c}$$

$$\Psi(X,\mu,s) = 0, \quad -1 \leq \mu < 0, s > 0, \tag{2d}$$

$$\Phi_c(x) = \int_{-1}^{1} \int_{0}^{\infty} \Psi(x,\mu,s)dsd\mu. \tag{2e}$$

Equations (2) are used to represent nonclassical transport problems on a homogeneous slab with azimuthal symmetry and vacuum boundaries. Here, Ψ is the nonclassical angular flux, X is the slab length, x is the

spatial variable, μ is the cosine of the polar angle, Q is an isotropic internal source, c is the scattering ratio, and Φ_c is the recovered scalar flux. Following [6], we define Ψ as

$$\Psi(x,\mu,s) \equiv e^{-\int_0^s \sigma_t(s')ds'} \sum_{m=0}^{M} \psi_m(x,\mu) L_m(s), \tag{3}$$

where ψ_m is the Laguerre moment of order m, M is the truncation order of the Laguerre expansion, and L_m is the m^{th} Laguerre polynomial.

We then carry out the following steps, summarized here due to space constraints: (i) introduce Eq. (3) into Eqs. (2) and perform a series of algebraic manipulations [6]; and (ii) apply the discrete ordinates formulation [1]. This allows us to write the following nonclassical spectral problem:

$$\mu_n \frac{d\psi_{m,n}(x)}{dx} + \psi_{m,n}(x) = S(x) + \frac{Q(x)}{2} - \sum_{j=0}^{m-1} \psi_{j,n}(x),\ 0 < x < X, \quad m = 0:M, n = 1:N, \tag{4a}$$

$$S(x) = \frac{c}{2} \sum_{n=1}^{N} \omega_n \sum_{k=0}^{M} \psi_{k,n}(x) \left[\int_0^\infty p(s) L_k(s) ds \right], \quad 0 < x < X, \tag{4b}$$

$$\psi_{m,n}(0) = 0, \quad m = 0:M, n = 1:\frac{N}{2}, \tag{4c}$$

$$\psi_{m,n}(X) = 0, \quad m = 0:M, n = \frac{N}{2}+1:N, \tag{4d}$$

$$\Phi_c(x) = \sum_{n=1}^{N} \omega_n \sum_{m=0}^{M} \psi_{m,n}(x) \int_0^\infty L_m(s) e^{-\int_0^s \sigma_t(s')ds'} ds. \tag{4e}$$

In Eqs. (4), the angular variable μ has been discretized in N discrete values μ_n; $\psi_{m,n}$ is the Laguerre moment of order m and direction n; ω_n is a weight of the Gauss-Legendre angular quadrature, and the scalar flux $\Phi_c(x)$ is recovered with Eq. (4e). Equations (4a) and (4b) are the one-speed, slab-geometry nonclassical spectral S_N equations. In [6], the authors lag the scattering source on the right-hand side of Eq. (4a), then numerically solve it using the conventional fine-mesh DD method with the SI scheme. Equations (4a) and (4b) in the SI scheme can be written as

$$\mu_n \frac{d\psi_{m,n}^{(l+1)}(x)}{dx} + \psi_{m,n}^{(l+1)}(x) = S^{(l)}(x) + \frac{Q(x)}{2} - \sum_{j=0}^{m-1} \psi_{j,n}^{(l+1)}(x), \tag{5a}$$

$$S^{(l)}(x) = \frac{c}{2} \sum_{n=1}^{N} \omega_n \sum_{k=0}^{M} \psi_{k,n}^{(l)}(x) \chi_k, \tag{5b}$$

$$\chi_k = \int_0^\infty p(s) L_k(s) ds, \tag{5c}$$

for $0 < x < X$, $m = 0:M$ and $n = 1:N$. Here, l is the iteration index.

Synthetic acceleration methods contain an error-correction step, which uses an approximation of the error equation to estimate the error at each iteration [7,8]. Consider an error in the $l+1$ iteration defined as

$$\epsilon_{m,n}^{(l+1)}(x) \equiv \psi_{m,n}(x) - \psi_{m,n}^{(l+1/2)}(x), \tag{6}$$

where $\psi_{m,n}(x)$ is the analytical solution of Eqs. (4) and $\psi_{m,n}^{(l+1/2)}(x)$ is the numerical solution of Eqs. (5) in the iteration $l+1/2$. Consequently, we define

$$S_{\epsilon}^{(l+1)}(x) \equiv S(x) - S^{(l+1/2)}(x) = \frac{c}{2}\sum_{n=1}^{N}\omega_n \sum_{k=0}^{M}\epsilon_{k,n}^{(l+1)}(x)\chi_k. \tag{7}$$

We write (5a) for the iteration $l+1/2$ and subtract this equation from Eq. (4a). Adding and subtracting the term $S^{l+1/2}(x)$ in the resulting equation, we combine the terms to obtain

$$\mu_n \frac{d\epsilon_{m,n}^{(l+1)}(x)}{dx} + \epsilon_{m,n}^{(l+1)}(x) = S_{\epsilon}^{(l+1)}(x) + S^{(l+1/2)}(x) - S^{(l)}(x) - \sum_{j=0}^{m-1}\epsilon_{j,n}^{(l+1)}(x), \tag{8}$$

for $0 < x < X$, $m = 0 : M$ and $n = 1 : N$. With this equation for the error, we can write a "general" standard synthetic acceleration method as

$$\mu_n \frac{d\psi_{m,n}^{(l+1/2)}(x)}{dx} + \psi_{m,n}^{(l+1/2)}(x) = S^{(l)}(x) + \frac{Q(x)}{2} - \sum_{j=0}^{m-1}\psi_{j,n}^{(l+1/2)}(x), \tag{9a}$$

$$S^{(l)}(x) = \frac{c}{2}\sum_{n=1}^{N}\omega_n \sum_{k=0}^{M}\psi_{k,n}^{(l)}(x)\chi_k, \tag{9b}$$

$$\mu_n \frac{d\epsilon_{m,n}^{(l+1)}(x)}{dx} + \epsilon_{m,n}^{(l+1)}(x) = S_{\epsilon}^{(l+1)}(x) + S^{(l+1/2)}(x) - S^{(l)}(x) - \sum_{j=0}^{m-1}\epsilon_{j,n}^{(l+1)}(x), \tag{9c}$$

$$S_{\epsilon}^{(l+1)}(x) = \frac{c}{2}\sum_{n=1}^{N}\omega_n \sum_{k=0}^{M}\epsilon_{k,n}^{(l+1)}(x)\chi_k, \tag{9d}$$

$$S^{(l+1)}(x) = S^{(l+1/2)}(x) + S_{\epsilon}^{(l+1)}(x), \tag{9e}$$

for $0 < x < X$, $m = 0 : M$ and $n = 1 : N$. Next, we briefly summarize the steps involved in the synthetic acceleration scheme.

For the first iteration, we solve Eq. (9a) with $S^{(0)}(x) = 0$ to calculate $\psi_{m,n}^{(1/2)}(x)$ for all values of m and n, which consequently yields $\Phi_c^{(1/2)}(x)$. For $m = 0$, the last term on the right-hand side of Eq. (9a) is zero. Thus, for $m > 0$, this term is known because we have already calculated the terms up to $m-1$ in the previous steps. After this, $S^{(1/2)}(x)$ is calculated by Eq. (9b). Then, instead of moving on to the next iteration as in the standard SI scheme, we numerically solve Eq. (9c) for $m = 0 : M$ and $n = 1 : N$ to determine $S_{\epsilon}^{(1)}(x)$, defined in Eq. (9d). Next, we use Eq. (9e) to "correct" the source term, obtaining $S^{(1)}(x)$. After this correction, we perform the next iteration in Eq. (9a), and all the process is repeated until a stopping criterion is satisfied.

In this general case, Eq. (9c) for $\epsilon_{m,n}(x)$ has the same form as the transport equation. Synthetic acceleration methods rely on choosing simplified models to approximate the solution of this error-problem. If we consider a S_2 model, we have the S_2SA method [8], and Eq. (9c) becomes

$$\pm\frac{1}{\sqrt{3}}\frac{d\epsilon_{m,\pm}^{(l+1)}(x)}{dx} + \epsilon_{m,\pm}^{(l+1)}(x) = \frac{c}{2}\sum_{k=0}^{M}[\epsilon_{k,+}^{(l+1)}(x) + \epsilon_{k,-}^{(l+1)}(x)]\chi_k + S^{(l+1/2)}(x) - S^{(l)}(x) - \sum_{j=0}^{m-1}\epsilon_{j,\pm}^{(l+1)}(x), \tag{10}$$

for $0 < x < X$, $m = 0 : M$ and $n = 1 : N$. In Eq. (10), $\epsilon_{k,+}^{(l+1)}$ and $\epsilon_{k,-}^{(l+1)}$ represent the error in the $l+1$ iteration concerning the Laguerre moment $\psi_{m,n}^{(1/2)}(x)$ in the positive and negative discrete directions, respectively, as generated by the S_2 formulation. Equation (10) is solved by lagging the first term on the right-hand side, then numerically solving it using the conventional fine-mesh DD method with the SI scheme.

Furthermore, if we consider a diffusion approximation for the error-problem, we have the P_1SA method, with Eqs. (9c) and (9d) being substituted by

$$\frac{d\Upsilon_m^{(l+1)}(x)}{dx} + \upsilon_m^{(l+1)}(x) = c\sum_{k=0}^{M} \upsilon_k^{(l+1)}(x)\chi_k + 2S^{(l+1/2)}(x) - 2S^{(l)}(x) - \sum_{j=0}^{m-1} \upsilon_j^{(l+1)}(x), \tag{11a}$$

$$\frac{1}{3}\frac{d\upsilon_m^{(l+1)}(x)}{dx} + \Upsilon_m^{(l+1)}(x) = -\sum_{j=0}^{m-1} \Upsilon_j^{(l+1)}(x), \tag{11b}$$

for $0 < x < X$, $m = 0 : M$ and $n = 1 : N$. Here, Υ_m represents the second order Legendre moment for $\epsilon_{m,n}^{l+1}(x)$, while υ_m represents the first order Legendre moment for $\epsilon_{m,n}^{l+1}(x)$. We numerically solve Eqs. (11) considering vacuum boundary conditions of the Mark's type [9], and substitute Eq. (9e) by

$$S^{(l+1)}(x) = S^{(l+1/2)}(x) + \frac{c}{2}\sum_{k=0}^{M} \upsilon_k^{(l+1)}(x)\chi_k\,. \tag{12}$$

3. FOURIER CONVERGENCE ANALYSIS

Let us consider the SI scheme. By taking $Q(x) = 0$ and considering a single Fourier error mode with arbitrary $-\infty < \lambda < \infty$, we have adapted the assumptions considered in [7] to

$$S^{(l)}(x) = w^l(\lambda)e^{i\lambda x}, \tag{13a}$$

$$\psi^{(l+1)}(x) = \alpha_{m,n}(\lambda)w^l(\lambda)e^{i\lambda x}. \tag{13b}$$

Substituting Eqs. (13) into Eqs. (5), we write

$$(\mu_n i\lambda + 1)\alpha_{m,n}(\lambda) = 1 - \sum_{j=0}^{m-1} \alpha_{j,n}(\lambda), \tag{14a}$$

$$w(\lambda) = \frac{c}{2}\sum_{n=1}^{N}\omega_n \sum_{k=0}^{M} \alpha_{k,n}(\lambda)\chi_k, \tag{14b}$$

for $m = 0 : M$ and $n = 1 : N$. Solving Eq. (14a) for $\alpha(\lambda)$ we obtain

$$\alpha_{m,n}(\lambda) = \sum_{j=0}^{m} \frac{(-1)^j \binom{m}{j}}{(\mu_n i\lambda + 1)^{j+1}},\ m = 0 : M,\ n = 1 : N. \tag{15}$$

Introducing Eq. (15) in Eq. (14b) we obtain

$$w(\lambda) = \frac{c}{2}\chi_0 I_1 + \frac{c}{2}\sum_{k=1}^{M}\chi_k \left[I_1 + \sum_{j=1}^{k}(-1)^j \binom{k}{j} I_{j+1} \right]\,, \tag{16}$$

where

$$I_{j+1} = \int_{-1}^{1} \frac{d\mu}{(\mu i\lambda + 1)^{j+1}} \approx \sum_{n=1}^{N} \frac{\omega_n}{(\mu_n i\lambda + 1)^{j+1}}\,,\ j \geq 0. \tag{17}$$

The integral in Eq. (17) has an analytical result for integer j. Hence, the analytical spectral radius SR is [7]

$$SR = \max_{-\infty<\lambda<\infty} w(\lambda). \tag{18}$$

Next, we consider the S_2SA approach. By taking $Q(x) = 0$ and considering a single Fourier error mode with arbitrary $-\infty < \lambda < \infty$, we have adapted the assumptions considered in [7] to

$$\psi_{m,n}^{(l+1/2)}(x) = w^l(\lambda)\alpha_{m,n}(\lambda)e^{i\lambda x}, \tag{19a}$$

$$S^{(l)}(x) = w^l(\lambda)e^{i\lambda x}, \tag{19b}$$

$$\epsilon_{m,\pm}^{(l+1)}(x) = w^l(\lambda)\gamma_{m,\pm}(\lambda)e^{i\lambda x}, \tag{19c}$$

$$S^{(l+1/2)}(x) = w^l(\lambda)\beta(\lambda)e^{i\lambda x}, \tag{19d}$$

$$S^{(l+1)}(x) = w^{l+1}(\lambda)e^{i\lambda x}, \tag{19e}$$

for $m = 0 : M$ and $n = 1 : N$. We have introduced Eqs. (19) into Eqs. (9), with Eq. (9c) replaced by Eq. (10), and numerically solving the resulting system in order to determine the analytical SR for known input data.

For the P_1SA method we have applied a similar methodology. For this case we consider

$$\psi_{m,n}^{(l+1/2)}(x) = w^l(\lambda)\alpha_{m,n}(\lambda)e^{i\lambda x}, \tag{20a}$$

$$S^{(l)}(x) = w^l(\lambda)e^{i\lambda x}, \tag{20b}$$

$$\upsilon_{m}^{(l+1)}(x) = w^l(\lambda)\gamma_{m}(\lambda)e^{i\lambda x}, \tag{20c}$$

$$\Upsilon_{m}^{(l+1)}(x) = w^l(\lambda)\rho_{m}(\lambda)e^{i\lambda x}, \tag{20d}$$

$$S^{(l+1/2)}(x) = w^l(\lambda)\beta(\lambda)e^{i\lambda x}, \tag{20e}$$

$$S^{(l+1)}(x) = w^{l+1}(\lambda)e^{i\lambda x}, \tag{20f}$$

for $m = 0 : M$ and $n = 1 : N$. We have introduced Eqs. (20) into Eqs. (9), with Eq. (9c) replaced by Eqs. (11), and numerically solved the resulting system in order to determine the analytical SR for known input data.

The computational code for these Fourier convergence analyses has c, M, and $p(s)$ as inputs and the analytical SR as output. In all cases, the numerical SR is estimated by [10]

$$SR_{Numerical} = \frac{\|\mathbf{\Phi}_c^{i+2} - \mathbf{\Phi}_c^{i+1}\|_2}{\|\mathbf{\Phi}_c^{i+1} - \mathbf{\Phi}_c^{i}\|_2}, \tag{21}$$

where $\mathbf{\Phi}_c$ is the vector of classical scalar fluxes.

4. NUMERICAL RESULTS

In order to illustrate the methodology presented in Section 2, we consider a slab-geometry model-problem with the following features: $X = 20\ cm$, $\sigma_t = 1.0\ cm^{-1}$, $N = 16$, 200 spatial discretization cells, vacuum boundary conditions, $Q(x) = 10\ cm^{-3}s^{-1}$ and the stopping criterion given by [8]

$$\frac{\|\mathbf{\Phi}_c^{l+1} - \mathbf{\Phi}_c^{l}\|_2}{\|\mathbf{\Phi}_c^{l}\|_2} \leq 10^{-6} = \zeta. \tag{22}$$

It has been shown [12] that certain diffusion-based approximations to the classical transport equation can be represented exactly by Eqs. (2a) and (2b) when the total cross-section $\sigma_t(s)$ (and hence $p(s)$) is appropriately chosen. That being the case, the collision rate density ($\sigma_t\Phi^{db}(x)$) of the diffusion-based approximations to the classical transport equation (e.g., simplified P_1 or simplified P_3 equations) should match the collision rate density generated by the solution of Eqs. (2). That is,

$$\sigma_t\Phi^{db}(x) = \int_0^{\infty} \sigma_t(s) \int_{-1}^{1} \Psi(x, \mu, s) d\mu ds. \tag{23}$$

Table 1: Parameters a, b, c and d [12].

Parameter	Value
a	5.642025
b	-2.941340
c	0.469086
d	-1.161256

Following the procedure described in Section 2, Eq. (23) can be approximated as

$$\sigma_t \Phi^{db}(x) = \sum_{n=1}^{N} \omega_n \sum_{k=0}^{M} \psi_{k,n}(x) \chi_k. \tag{24}$$

Still considering the reference [12], if we assume the free-path distribution function as $p(s) = 3\sigma_t^2 s e^{-\sqrt{3}\sigma_t s}$ in the solution of Eqs. (2), the right-hand side of Eq. (24) should match the collision-rate density of the classical SP_1 model. In a similar fashion, if we assume the free-path distribution function as $p(s) = s\sigma_t^2(ae^{b\sigma_t s} + ce^{d\sigma_t s})$ in the solution of Eqs. (2), the right-hand side of Eq. (24) should match the collision-rate density of the classical SP_3 model. The values of a, b, c and d are given in Table 1.

Tables 2 and 3 illustrate the applicability of the nonclassical model for nonexponential $p(s)$, with the numerical results obtained being very close to the SP_1 and SP_3 reference results [13].

Table 2: Collision-rate density for $p(s) = 3\sigma_t^2 s e^{-\sqrt{3}\sigma_t s}$.

$x\ (cm)$	SP_1[13]	P_1SA			Relative error (%)		
		$M = 10$	$M = 50$	$M = 70$	$M = 10$	$M = 50$	$M = 70$
	$c = 0.0$						
2	9.8434E0	9.8448E0	9.8448E0	9.8448E0	1.422E-2	1.422E-2	1.422E-2
10	9.9999E0	1.0000E1	9.9999E0	9.9999E0	1.000E-3	0.0000E0	0.0000E0
18	9.8434E0	9.8448E0	9.8448E0	9.8448E0	1.422E-2	1.422E-2	1.422E-2
	$c = 0.5$						
2	1.8988E1	1.8991E1	1.8991E1	1.8991E1	1.579E-2	1.579E-2	1.579E-2
10	1.9999E1	1.9999E1	1.9999E1	1.9999E1	0.0000E0	0.0000E0	0.0000E0
18	1.8988E1	1.8991E1	1.8991E1	1.8991E1	1.579E-2	1.579E-2	1.579E-2
	$c = 0.9$						
2	7.4591E1	7.4587E1	7.4587E1	7.4587E1	5.362E-3	5.362E-3	5.362E-3
10	9.9364E1	9.9365E1	9.9365E1	9.9365E1	1.006E-3	1.006E-3	1.006E-3
18	7.4591E1	7.4587E1	7.4587E1	7.4587E1	5.362E-3	5.362E-3	5.362E-3

Tables 4 and 5 display the analytical and numerical results obtained for the spectral radii SR with the same two choices of $p(s)$, according to the methodology described in Section 3 and Eq. (21). For numerical SR calculations, we consider a homogeneous problem with $\sigma_t = 1.0\ cm^{-1}$, $X = 100\ cm$, $N = 32$, 10000 spatial discretization cells, vacuum boundaries, $\zeta = 10^{-12}$, $Q(x) = 10\ cm^{-3}s^{-1}$ (adapted from [8]). As we can see in these tables, the synthetic acceleration methods are efficient in reducing the number of iterations required to achieve convergence. This is a consequence of the decrease in the SR.

Table 3: Collision-rate density for $p(s) = s\sigma_t^2(ae^{b\sigma_t s} + ce^{d\sigma_t s})$.

$x\ (cm)$	SP_3[13]	P_1SA			Relative error (%)		
		$M = 10$	$M = 50$	$M = 70$	$M = 10$	$M = 50$	$M = 70$
	$c = 0.0$						
2	9.8204E0	9.8190E0	9.8212E0	9.8212E0	1.425E-2	8.146E-3	8.146E-3
10	9.9999E0	9.9999E0	9.9999E0	9.9999E0	0.0000E0	0.0000E0	0.0000E0
18	9.8204E0	9.8190E0	9.8212E0	9.8212E0	1.425E-2	8.146E-3	8.146E-3
	$c = 0.5$						
2	1.9044E1	1.9042E1	1.9046E1	1.9046E1	1.050E-2	1.050E-2	1.050E-2
10	1.9999E1	1.9999E1	1.9999E1	1.9999E1	0.0000E0	0.0000E0	0.0000E0
18	1.9044E1	1.9042E1	1.9046E1	1.9046E1	1.050E-2	1.050E-2	1.050E-2
	$c = 0.9$						
2	7.6388E1	7.6380E1	7.6386E1	7.6386E1	1.047E-2	2.618E-3	2.618E-3
10	9.9299E1	9.9299E1	9.9299E1	9.9299E1	0.0000E0	0.0000E0	0.0000E0
18	7.6388E1	7.6380E1	7.6386E1	7.6386E1	1.047E-2	2.618E-3	2.618E-3

5. DISCUSSION

As we can see in Tables 2 and 3, the nonclassical model correctly reproduces the reference SP_1 and SP_3 results for this model-problem when considering the appropriate $p(s)$. However, depending on the $p(s)$ choice, the improper integral in Eq. (5c) may not converge. In this case, it is not possible to obtain numerical results by performing the methodology proposed. We intend to test different $p(s)$ and parameters σ_t in order to investigate the improper integral's behavior. Notice that, for the fine mesh considered in the model-problem presented, there was not a significant advantage in increasing the truncation order M to values higher than ten. In future work, we intend to investigate the performance for a coarser mesh and to expand the methodology to an energy multigroup formulation.

Table 4: SR for $p(s) = 3\sigma_t^2 s e^{-\sqrt{3}\sigma_t s}$ and $M = 70$.

c	Number of iterations			Numerical SR			Analytical SR		
	SI	S_2SA	P_1SA	SI	S_2SA	P_1SA	SI	S_2SA	P_1SA
0.8	117	14	15	0.7997	0.1993	0.1994	0.8000	0.2005	0.2005
0.9	240	15	16	0.8996	0.2226	0.2233	0.9000	0.2251	0.2251
0.99	2220	15	16	0.9897	0.2434	0.2443	0.9900	0.2475	0.2475
0.999	15902	15	16	0.9986	0.2452	0.2461	0.9990	0.2497	0.2497
0.9999	47299	15	16	0.9996	0.2456	0.2471	0.9999	0.2499	0.2499

According to Tables 4 and 5, for the model-problem presented, we can conclude that the P_1SA and the S_2SA are efficient in reducing the number of iterations required to achieve a converged solution. The advantage of synthetic acceleration methods is clear when $c \approx 1$, since the number of iterations required in the standard SI for this case is considerably higher when compared to the number of iterations required in the synthetic acceleration methods. With the Fourier convergence analysis proposed, we can predict the

Table 5: SR for $p(s) = s\sigma_t^2(ae^{b\sigma_t s} + ce^{d\sigma_t s})$ and $M = 70$.

c	Number of iterations			Numerical SR			Analytical SR		
	SI	S_2SA	P_1SA	SI	S_2SA	P_1SA	SI	S_2SA	P_1SA
0.8	117	13	14	0.7997	0.1623	0.1674	0.8000	0.1693	0.1693
0.9	240	14	15	0.8996	0.1865	0.1877	0.9000	0.1911	0.1911
0.99	2220	14	15	0.9897	0.2097	0.2099	0.9900	0.2114	0.2114
0.999	15902	14	15	0.9986	0.2104	0.2115	0.9990	0.2135	0.2135
0.9999	47299	14	15	0.9996	0.2117	0.2119	0.9999	0.2137	0.2137

iterative methods' performance, which is very useful upon developing new iterative schemes. The increase of truncation order M to values higher than ten has not significantly affected the results. As future work, we intend to investigate the methods' performance when the cell's optical thickness increases, analyzing the numerical SR as a function of the optical thickness. In both the S_2SA and P_1SA approaches presented here, we considered the same mesh for the main problem and the error problem. Thus, we intend to test problems using a coarser mesh for the error-problem.

ACKNOWLEDGEMENTS

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. The second and third authors acknowledge the financial support of Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ).

REFERENCES

[1] J.J. Duderstadt and W.R. Martin, *Transport Theory*, Wiley-Interscience, New York, USA (1979).

[2] A.B. Davis and F. Xu, "A Generalized Linear Transport Model for Spatially Correlated Stochastic Media", *Journal of Computational and Theoretical Transport*, **43**, 474–514, 2014.

[3] R. Vasques and E.W. Larsen, "Non-classical particle transport with angular-dependent path-length distributions. II: Application to pebble bed reactor cores", *Annals of Nuclear Energy*, **70**, 301–311, 2014.

[4] B. Bitterli, S. Ravichandran, T. Müller, M. Wrenninge, J. Novák, S. Marschner, and W. Jarosz, "A Radiative Transfer Framework for Non-exponential Media", *In: ACM Transactions on Graphics*, 37.6 (November 2018).

[5] E.W. Larsen and R. Vasques, "A Generalized Linear Boltzmann Equation for Non-Classical Particle Transport" *Journal of Quantitative Spectroscopy & Radiative Transfer*, **112**, 619-631, 2011.

[6] R. Vasques, L.R.C. Moraes, R.C. Barros, and R.N. Slaybaugh, "A Spectral Approach for Solving the Nonclassical Transport Equation", *Journal of Computational Physics*, **402**, 109078, 2020.

[7] M.L. Adams and E.W. Larsen, "Fast Iterative Methods for Discrete-Ordinates Particle Transport Calculations", *Progress in Nuclear Energy*, **40**, 3-159, 2002.

[8] J.K. Patel, L.R.C. Moraes, R. Vasques, and R.C. Barros, "Transport Synthetic Acceleration for the Solution of the One-Speed Nonclassical Spectral S_N Equations in Slab Geometry", *Journal of Computational and Applied Mathematics*, **401**, 113768, 2022.

[9] G.I. Bell and S. Glasstone, *Nuclear Reactor Theory*, Van Nostrand Reinhold, New York, USA (1970).

[10] E.W. Larsen and J.E. Morel, "Advances in Discrete-Ordinates Methodology", *In: Nuclear Computational Science: A Century in Review*, Springer Netherlands, Dordrecht, pp. 1-84 (2010).

[11] R.G. McClarren, "Theoretical Aspects of the Simplified P_n Equations", *Transport Theory and Statistical Physics*, **39**, 73–109, 2010.

[12] M. Frank, K. Krycki, E.W. Larsen, and R. Vasques, "The Non-Classical Boltzmann Equation and Diffusion-Based Approximations to the Boltzmann Equation", *SIAM Journal of Applied Mathematics*, **75**, 1329-1345, 2015.

[13] A.S. Silva, *Métodos sintéticos analíticos de solução da equação de transporte de nêutrons com aproximações da teoria P_N*, Master's thesis, Universidade do Estado do Rio de Janeiro (2019).